Maximizing Induced Cardinality Under a Determinantal Point Process

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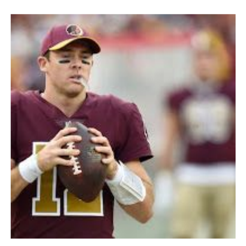
Motivation

Diversity can be useful for recommender systems, for two main reasons:

- **Uncertainty** --- search engine query "java" has multiple interpretations


- **Exploration** --- news feed contents should span topics of user interest


Sports


Technology


Politics


Business

Determinantal Point Processes (DPPs)

DPPs are a means of trading off item **quality** with **diversity**. A DPP over n items is parameterized by an n-by-n matrix L whose diagonal captures item quality and whose off-diagonal captures item-item similarity.

Example --- Game app recommendation:

$$L = \begin{bmatrix} 2.5 & 2.3 & 0.5 & 0.2 & 0.2 \\ 2.3 & 2.4 & 0.4 & 0.2 & 0.2 \\ 0.5 & 0.4 & 2.0 & 0.1 & 0.1 \\ 0.2 & 0.2 & 0.1 & 1.5 & 1.4 \\ 0.2 & 0.2 & 0.1 & 1.4 & 1.4 \end{bmatrix}$$

Probability of a set:

$$\mathcal{P}_L(S) \propto \det(L_S)$$

Example:

quality - similarity

$$\det(L_{1,2}) = L_{11}L_{22} - L_{12}^2 = 2.5 * 2.4 - 2.3^2$$

Highest-probability set:

$$\max_{S:|S|=3} \det(L_S) =$$

Training a DPP Recommender System

Goal --- Recommend k items from a much larger set of n items.

Training data --- r previously-recommended k-sets: $[S_1, S_2, \dots, S_r]$ and resulting user engagement sets: $[E_1, E_2, \dots, E_r]$ (e.g., which items a user clicked on, or watched, or read, etc.).

Likelihood objective --- Modeling user behavior as a DPP, maximize probability of engaged sets by optimizing parameters θ that define L.

$$\max_\theta \sum_{i=1}^r \log(\mathcal{P}_{L^{(i)}(\theta)}(E_i))$$

$$E_i \subseteq S_i$$

$$|S_i| \times |S_i| \text{ matrix}$$

Generating Recommendations

Standard inference-time objective --- Maximum a posteriori (**MAP**):

MAP $\quad \max_{S:|S|=k} \mathcal{P}_L(S) = \max_{S:|S|=k} \det(L_S)$

Mis-match --- Training modeled *engaged-with* items as draws from a DPP, not the set of all *recommended* items. Hence, this MAP objective really represents the probability that a user will engage with *every* item in S.

More natural goal --- Recommend S that maximizes expected cardinality of the induced engagements E; maximum induced cardinality (**MIC**):

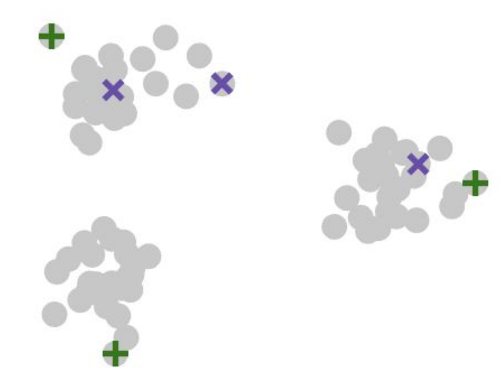
MIC $\quad \max_{S:|S|=k} \sum_{E \subseteq S} |E| \mathcal{P}_{L_S}(E)$

Main contribution of this work --- Proposal and analysis of MIC.

MAP Failure Case

Low rank kernels --- If rank(L) < k, then **MAP** has equal value (zero) for all size-k sets. **MIC** on the other hand differentiates among k-sets.

Example --- Each item is represented by a 2-dimensional feature vector and data forms 3 clusters. **MIC** selects one item in each cluster, while **MAP** selects 3 items at random.



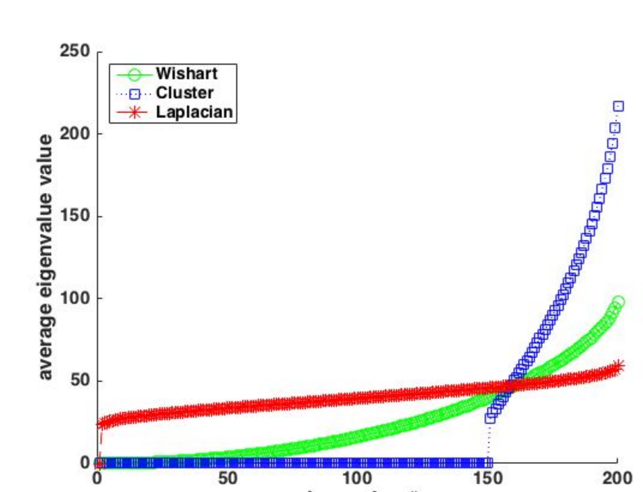
Properties of Induced Cardinality

- Computable in $O(k^3)$ time:

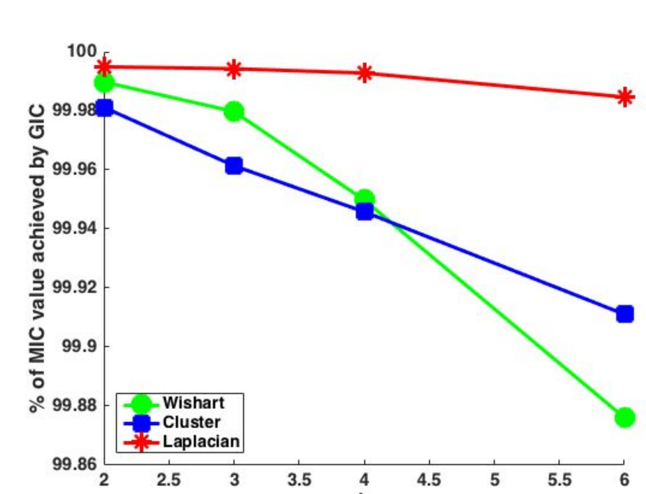
$$f(S) = \sum_{E \subseteq S} |E| \mathcal{P}_{L_S}(E) = \boxed{\mathrm{Tr}(I - (L_S + I)^{-1})}$$

- Monotone increasing and fractionally subadditive
- Submodular if L is an M-matrix (all off-diagonal entries are non-positive)
- NP-hard to maximize

Direct Optimization



Kernel matrix types --- Experimented with three types of L matrices, each with a distinct spectrum: **Wishart**, **cluster** (n items divided into k Gaussian clusters), and **graph Laplacian** (n-node graph, Erdos-Renyi model with edge existence parameter p = 0.2).



Small kernel: n = 12
MIC --- Exact max.
GIC --- Greedy algorithm on f. No approximation guarantees in general, but performs well in practice. Best on Laplacians (which are M-matrices), and achieves more than 99% of maximum possible value for other kernels.

Series Approximation

Geometric series representation ---

- Define: $m = \lambda_n(L) + 1$ and $B = (m-1)I - L$
- Then using the Neumann series representation of the matrix inverse:

$$f(S) = |S| - \sum_{i=0}^{\infty} \frac{\mathrm{Tr}(B_S^i)}{m^{i+1}}$$

- The first few terms are a **monotone submodular approximation**:

$$\boxed{\hat{f}(S) = |S| - \frac{|S|}{m} - \frac{\mathrm{Tr}(B_S)}{m^2} - \frac{\mathrm{Tr}(B_S^2)}{m^3}}$$

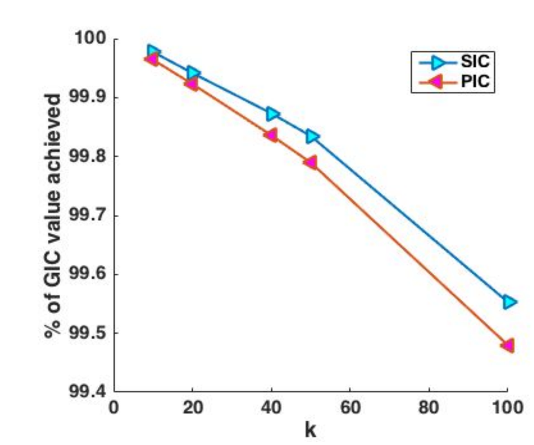
Goodness of approximation --- For all sets S of size k:

$$\frac{f(S)}{\hat{f}(S)} \geq 1 - \frac{mr_3}{(m-1)k - r_1 - r_2}, \quad \text{with } r_i = \sum_{j=n-k+1}^{n} \left(\frac{\lambda_j(B)}{m}\right)^i$$

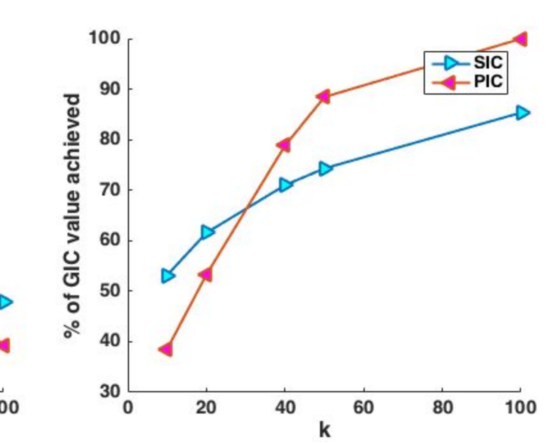
Best when smaller eigenvalues of L are close to $\lambda_n(L)$.

Optimization of Approximations

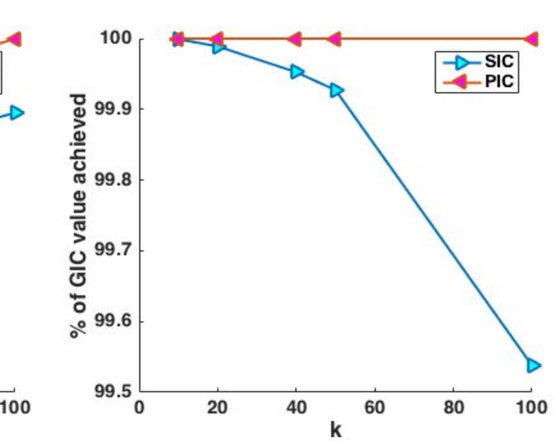
Kernel size: n = 200.
PIC --- Greedy algorithm on f after projecting L to an M-matrix.
SIC --- Greedy algorithm on the (submodular) series approximation.



Wishart kernels

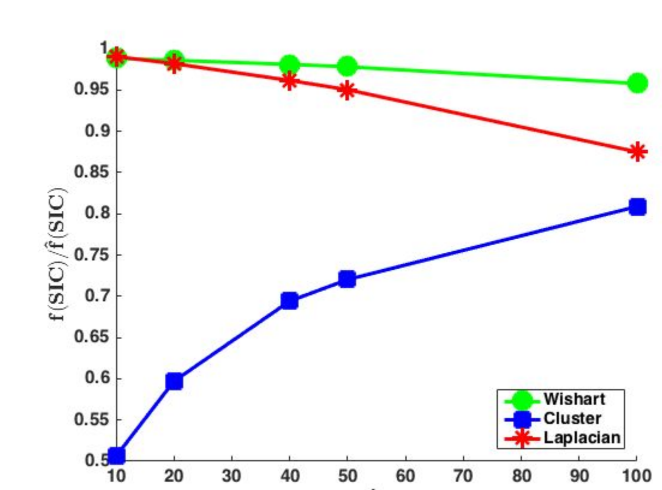


Cluster kernels



Laplacian kernels

PIC performance --- PIC does well when the projection to M-matrix does not alter the objective too much; graph Laplacian kernels are already M-matrices, so PIC is equivalent to GIC in the third graph.



SIC performance --- SIC does well for Wishart and Laplacian kernels, but struggles with the cluster kernels. This is because the $f/\hat{f}$ ratio decays slowly with k for Wishart and Laplacian, but grows dramatically with k for cluster kernels. (See eigenvalue plot.)`

Runtime --- GIC (and PIC, ignoring the initial projection step) are $O(nk^3)$ while SIC is a factor of k faster. For n = 500 and k = 250, SIC runs about 18 times faster than GIC.

Conclusion --- Use SIC when speed is important, or when approximation guarantee is required.