Maximizing Induced Cardinality Under a Determinantal Point Process

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Motivation

Diversity can be useful for recommender systems, for two main reasons:

- Uncertainty — search engine query "java" has multiple interpretations
- Exploration — news feed contents should span topics of user interest

Determinantal Point Processes (DPPs)

DPPs are a means of trading off item quality with diversity. A DPP over n items is parameterized by an n-by-n matrix L whose diagonal captures item quality and whose off-diagonal captures item-item similarity.

Example — Game app recommendation:

<table>
<thead>
<tr>
<th>Item</th>
<th>Sports</th>
<th>Technology</th>
<th>Politics</th>
<th>Business</th>
</tr>
</thead>
<tbody>
<tr>
<td>Value</td>
<td>2.5</td>
<td>0.2</td>
<td>0.2</td>
<td>0.2</td>
</tr>
<tr>
<td>Value</td>
<td>2.3</td>
<td>0.4</td>
<td>0.2</td>
<td>0.2</td>
</tr>
<tr>
<td>Value</td>
<td>0.5</td>
<td>0.1</td>
<td>1.5</td>
<td>1.4</td>
</tr>
<tr>
<td>Value</td>
<td>0.2</td>
<td>0.2</td>
<td>1.4</td>
<td>1.4</td>
</tr>
</tbody>
</table>

Probability of a set: \(P_L(S) \propto \det(L_S)\)

Example:

- Quality-similarity: \(\det(L_{1,2}) = L_{11}L_{22} - L_{12}L_{21}\)
- Highest-probability set: \(\max_{S:|S|=3} \det(L_S) = \{\text{Item} 1, \text{Item} 2, \text{Item} 3\}\)

Training a DPP Recommender System

Goal — Recommend k items from a much larger set of n items.

Training data — previously-recommended k-sets: \([S_1, S_2, \ldots, S_k]\)

and resulting user engagement sets: \([E_1, E_2, \ldots, E_k]\)

(e.g., which items a user clicked on, or watched, or read, etc.).

Likelihood objective — Modeling user behavior as a DPP, maximize probability of engaged sets by optimizing parameters \(\lambda\) that define L.

\[
\max_{\lambda} \sum_{i=1}^{n} \log(P(E_i | \lambda)) \quad |S_i| \times |S_i| \text{ matrix}
\]

Generating Recommendations

Standard inference-time objective — Maximum a posteriori (MAP):

\[\text{MAP} = \max_{\lambda} P_L(S) = \max_{\lambda} \det(L_S)\]

Dis-claimed — Training modeled engaged-with items as draws from a DPP, not the set of all recommended items. Hence, this MAP objective really represents the probability that a user will engage with every item in S.

More natural goal — Recommend S that maximizes expected cardinality of the induced engagements E; maximum induced cardinality (MIC):

\[\text{MIC} = \max_{|S|=k} \sum_{E \subseteq S} |E|P_L(E)\]

Main contribution of this work — Proposal and analysis of MIC.

MAP Failure Case

Low rank kernels — If rank(L) ≤ k, then MAP has equal value (zero) for all size-k sets.

MIC on the other hand differentiates among k-sets.

Example — Each item is represented by a 2-dimensional feature vector and data forms 3 clusters. MIC selects one item in each cluster, while MAP selects 3 items at random.

Properties of Induced Cardinality

- Computable in \(O(k^3)\) time:

\[f(S) = \sum_{E \subseteq S} |E|P_L(E) = \frac{\text{Tr}(I - (L_L + I)^{-1})}{m}\]

- Monotone increasing and fractionally subadditive
- Submodular if L is an M-matrix (all off-diagonal entries are non-positive)
- NP-hard to maximize

Direct Optimization

Kernel matrix types — Experimented with three types of L matrices, each with a distinct spectrum: Wishart, cluster (n items divided into k Gaussian clusters), and graph Laplacian (n-node graph, Erdos-Renyi model with edge existence parameter \(p = 0.2\)).

Small kernel: \(n = 12\)

MIC — Exact max.

GIC — Greedy algorithm on f. No approximation guarantees in general, but performs well in practice. Best on Laplacians (which are M-matrices), and achieves more than 99% of maximum possible value for other kernels.

GIC performance — SIC does well when the projection to M-matrix does not alter the objective too much, graph Laplacian kernels are already M-matrices, so PIC is equivalent to GIC in the third graph.

SIC performance — SIC does well for Wishart and Laplacian kernels, but struggles with the cluster kernels. This is because the \(f/f^\star\) ratio decays slowly with k for Wishart and Laplacian, but grows dramatically with k for cluster kernels. (See eigenvalue plot.)

Series Approximation

Geometric series representation —

- Define: \(m = \lambda_n(L) + 1\) and \(B = (m - 1)I - L\)
- Then using the Neumann series representation of the matrix inverse:

\[f(S) = |S| \sum_{i=0}^{\infty} \frac{\text{Tr}(B_L^i)}{m^{i+1}}\]

- The first few terms are a monotone submodular approximation:

\[f(S) = |S| \frac{\text{Tr}(B_L) - \text{Tr}(B_L^2)}{m^2}\]

Goodness of approximation — For all sets S of size k:

\[f(S) \geq 1 - \frac{m+3}{(m-1)k - r_1 - r_2} \text{ with } r_j = \sum_{j=m-k+1}^{n} \left( \frac{\lambda_j(B)}{m} \right)^j\]

Best when smaller eigenvalues of L are close to \(\lambda_n(L)\).

Optimization of Approximations

Kernel size: \(n = 200\).

PIC — Greedy algorithm on f after projecting L to an M-matrix.

SIC — Greedy algorithm on the (submodular) series approximation.

PIC performance — PIC does well when the projection to M-matrix does not alter the objective too much, graph Laplacian kernels are already M-matrices, so PIC is equivalent to GIC in the third graph.

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Runtime — GIC (and PIC, ignoring the initial projection step) are \(O(nk^3)\) while SIC is a factor of k faster. For \(n = 500\) and \(k = 250\), SIC runs about 18 times faster than GIC.

Conclusion — Use SIC when speed is important, or when approximation guarantee is required.